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Fermi transport and Weylian electromagnetism

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Abstract. *This article describes a class of geometric structures on the cotangent bundle of space-time in which the vector potential of the electromagnetic field is related to the dilation 1-form of a semi-metric connection. These geometries satisfy a set of postulates that globalize the postulates of special relativity. When the field strength vanishes the resulting geometric structure realizes Fermi transport as parallel translation relative to a connection on the cotangent bundle.*

This article describes an approach to classical electromagnetism that is in spirit similar to the description given by H. Weyl; see [5]. Weyl's treatment of electromagnetism inspired the development of gauge theory in quantum mechanics, but as a classical theory it was flawed. The difficulties with Weyl's theory can be traced to the fact that it does not provide a natural representation of the Lorentz force law. In Weylian electrodynamics the vector potential is represented as the dilation of a semimetric connection; and consequently the vector potential enters into the mechanical equations of motion. Recall that a connection ∇ is *semimetric* for a metric g if there is a 1-form λ called the *dilation* of ∇ such that $\nabla g = \lambda \otimes g$. This article shall show that when space time geometry is lifted to the cotangent bundle a new structure emerges in which the vector potential appears as an dilation, but in a way that is consistent with the Lorentz force law.

The construction that accomplishes this is based upon an extension of special relativ-

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ity to manifolds. In general relativity, special relativistic effects are partially modeled by Fermi transport. Using techniques developed in [3], we show that on the cotangent bundle of space-time there is a natural almost Kahler structure from which Fermi transport can be recovered. Structures that give Fermi transport can be characterized intrinsically by two conditions that are extensions of the two axioms of special relativity; namely, the equivalence of Galilean frames and the constancy of the speed of light. Just as in Weyl's construction, it is the second axiom that allows the vector potential to be represented as a dilation. The requirement that the speed of light is the same in all frames is equivalent to the condition that all frames possess the same light cone. This condition is most easily implemented by the use of semimetric connections. Just as in Weyl's theory, the dilation of these connections can be related to the vector potential, however in this contructions the vector potential is also related to its standard representation in mechanics as an affine translation.

This article is divided into three parts. The first section reviews material from [3] needed in this construction. In the second section we show how geometric structures of section 1 can be used to realize Fermi transport in terms of connections on the cotangent bundle. In the last section we study these connections intrinsically and give a characterization of «Fermi like» connections that lead to a formulation of electromagnetism in terms of semimetric geometry. Finally, I would like to thank Blake Temple for many conversations that helped in the exposition of this material.

SECTION 1. NONLINEAR GEOMETRIES

We use the following notation. Let M be a C^∞-manifold. Let $\mathcal{F}(M)$ be the ring of germs of C^∞-functions on M. Denote the module of $C^\infty - (p,q)$-tensor on M by $\mathcal{T}^{(p,q)}(M)$. Denote the module of smooth vector fields on M by $\mathcal{X}(M)$, and the module of q-forms on M by $\mathcal{E}^q(M)$. If X is a subbundle of TM, let $\mathcal{I}^{(p,q)}(X)$ be the (p,q)-tensors which when viewed as q-linear maps take their values in the p-fold tensor product of X. In particular, denote the vector fields with values in X by $\mathcal{X}(X)$. If $f \in \mathcal{F}(M)$, then f *is constant along* X if $Vf = 0$ for all $V \in \mathcal{X}(X)$.

This section presents the results of [3] needed in the present construction. This is done for the sake of completenes, and because the results of [3] are not widely known. The discussion of [3] concerns the notion of a non-linear geometry. These structures are natural extensions of the geometric structure determined by a linear connection on the cotangent bundle. Let M be an even dimensional smooth manifold. A *nonlinear geometry* on M is given by a triple $((X,Y),g,\omega)$. Here ω is a symplectic 2-form on M. X and Y are complementary Lagrangian subbundles of TM, and g is a metric defined only along X. In the following it shall always be assumed that X is integrable.

EXAMPLE 1.1. The example that motivates this definition is obtained when $M = T^\star N$

and ω is the canonical symplectic form. X is the vertical bundle VT^*N. If q is metric on N, then g is defined along X by affine translation of q. Y is chosen to be the horizontal distribution of the Levi-Civita connection; although in the following applications any horizontal Lagrangian distribution that contains the metric Hamiltonian vector field will suffice.

A triple $((X,Y),g,\omega)$ determines a pair of (1,1)-tensors on M. Most obvious is the projection $P \in \mathcal{I}^{(1,1)}(Y)$. Denote the complement of P by $P^\perp \in \mathcal{F}^{(1,1)}(X)$. Note that, since X and Y are Lagrangian, P has the property that $i(P)\omega = \omega$. Next, there is an almost complex structure $J \in \mathcal{I}^{(1,1)}(M)$ defined as follows. For $U \in \mathcal{X}(X)$ define $JU \in \mathcal{X}(Y)$ so that $g(V,U) = \omega(V,JU)$ for all $V \in \mathcal{X}(X)$. For $U \in \mathcal{X}(Y)$ define $JU \in \mathcal{X}(X)$ so that $g(JU,V) = \omega(U,V)$ for all $V \in \mathcal{X}(X)$. Using J, the metric g can be extended to TM by requiring that for $U,V \in \mathcal{X}(M) g(U,V) = \omega(U,JV)$. It is easy to see that g is an almost-Kahler metric and ω is the corresponding almost-Kahler form.

A non-linear geometry determines a pair of connections on M. Both connections are defined in terms of a pair of conntions along a distribution. A *connection along a distribution* X is an $\mathbb{R}$-linear map $D : \mathcal{X}(X) \times \mathcal{X}(X) \to \mathcal{X}(X)$ that is linear over $\mathcal{F}(M)$ in the first entry and satisfies the usual derivation property in the second. If Y is complementary to X, then D is $Y-symmetric$ if for $U,V \in \mathcal{X}(X)$ $D_U V - D_V U - P^\perp[U,V] = 0$. The following propositions define a pair of connections determined by a triple $((X,Y),g,\omega)$.

PROPOSITION 1.1. *If* $\bar\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ *defined by*

(i) for $U \in \mathcal{X}(X)$ *and* $V \in \mathcal{X}(Y), \bar\nabla_U V = P^\perp[U,V]$ *and* $\bar\nabla_U V = P[U,V]$, *and*

(ii) for $U,V \in \mathcal{X}(X)$ *(or* $\mathcal{X}(Y)$*) if* $i(Z)\omega = L_U i(V)\omega$ *then* $\bar\nabla_U V = P^\perp Z$ $(\text{or} PZ)$, *then* $\bar\nabla$ *is a symmetric connection on* M *that satisfies* $\bar\nabla P = \bar\nabla\omega = 0$. *Further, if* $X(\text{or}Y)$ *is integrable the* $\bar\nabla$ *is flat along* $X(\text{or}Y)$.

PROPOSITION 1.2. *Let* $D(\text{or}D')$ *be Levi-Civita connection for* g *along* $X(\text{or}Y)$. *If* $\bar\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ *is defined by*

(i) for $U,V \in \mathcal{X}(X)(\text{or}\mathcal{X}(Y))$ *let* $\nabla_U V = D_U V(D'_U V)$, *and*

(ii) for $U \in \mathcal{X}(X)$ *and* $V \in \mathcal{X}(Y)$ *let* $\nabla_U V = JD_U JV$ *and* $\nabla_V U = JD'_U JU$. *then* ∇ *is a connection on* M *that satisfies* $\nabla P = \nabla J = 0$ *and* $\nabla\omega = 0$. ■

The connection $\bar\nabla$ is generally known as the *Bott connection* and was first used by H. Hess in [2]. In the case where J is integrable, the connection ∇ has a torsion tensor of type (1,1) and is therefore orthogonal to the hermitian connection. In the following ∇ is called the *inertial connection* determined by $((X,Y),g,\omega)$.

An important characteristic of the inertial connection is that the torsion tensor determines the curvature. If R and T are the torsion and curvature of a connection D, recall that the first Bianchi identity states that $\mathrm{Alt}(R) = \mathrm{d}T$, where Alt is the skew symmetrization operator and d is the exterior derivative defined by D on vector valued forms. The inertial connection satisfies a stronger set of conditions.

PROPOSITION 1.3. (i) *For* $U, V \in \mathcal{X}(X)$ *and* $Z, W \in \mathcal{X}(M)$

$$\omega(R(U,V)Z,W) = \omega(\mathrm{d}T(U,V,PZ),P^{\perp}W) + \omega(\mathrm{d}T(U,V,PW),P^{\perp}Z).$$

(ii) *For* $U, V \in \mathcal{X}(Y)$ *and* $Z, W \in \mathcal{X}(M)$

$$\omega(R(U,V)Z,W) = \omega(\mathrm{d}T(U,V,P^{\perp}Z),PW) + \omega(\mathrm{d}T(U,V,P^{\perp}W),PZ).$$

■

The other components of the curvature are also determined by $\mathrm{d}T$, but we shall not need these results here; (see [3]).

Another useful property of an inertial connection is that the torsion determines the difference tensor $S = \bar{\nabla} - \nabla$. If $U \in \mathcal{X}(Y)$ and $V \in \mathcal{X}(X)$, then from the definitions of $\bar{\nabla}$ and T it is clear that $S(U,V) = -P^{\perp}T(U,V)$ and $S(V,U) = -PT(V,U)$. The next proposition shows that T also determines the value of S along X and Y.

PROPOSITION 1.4. (i) *For* $U, V, W \in \mathcal{X}(X)\omega(W, PT(V, JV)) = g(U, JS(V, W))$, *and*

(ii) For $U, V, W \in \mathcal{X}(Y)\omega(W, P^{\perp}T(V, JV)) = g(U, S(V, W))$. ■

Nonlinear geometries are useful in relativity because they provide a geometric calculus for frames of observers of a space time N. Although a frame is usually represented by a time-like vector field, it is more natural in this context to take the dual point of view and identify a frame with a field of time-like 1-forms λ. For each $p \in N, \lambda_p$ is the clock determined by an observer of the frame at p. A curve γ is parameterized by the frame's clock's if $\lambda(\gamma) = 1$. If $\mathrm{d}\lambda = 0$ the frames is said to be *synchronous*. A locally defined function φ that satisfies $\mathrm{d}\varphi = \lambda$ is a time function for a synchronous frame since if γ is parameterized by the frame, then $\varphi(\gamma(t)) = t + a$. To investigate the special relativistic properties of frames in the setting of general relativity consider a class of frames $\mathcal{O}$ that has the property that for each $p \in N$ and any time-like 1-form $\lambda \in T^*N_p$ there is a unique $\mu \in \mathcal{O}$ such that $\mu_p = \lambda$. Such a set of frames is said to be *complete*. The standard example of a complete set of frames is the set of Galilean frames given by the time-like constant differential 1-forms on Minkowski space. Since differential 1-forms determine submanifolds of T^*N that are transverse to the vertical, one notes that a complete set of frames determines a local foliation of the light cone that is transverse to the vertical.

To apply the geometric construction of this section to relativity, the symplectic manifold M will be interpreted as the space of observers, for a space time $N, M \subset T^*N$. If the distribution Y is integrable, the leaves of Y represent a complete set of frames. The condition that Y be Lagrangian implies that Y determines a complete set of synchronous frames. The leaves of X decompose M into classes of simultaneous observers. For a space-time $N, X = VT^*N$. The advantage of this point of view is that by using the Bott connection one can associate an affine structure with any complete set of synchronous frames. This affine structure agrees with the coordinate affine structure when the complete set is a set of time-like 1-forms that are constant in some coordinate system.

SECTION 2. FERMI TRASPORT

Let N be a Lorentzian manifold with a Lorentzian metric q. Let ∇^L be the Levi-Civita connection on N. If $\gamma : \mathbf{R} \to N$ is a non-lightlike curve, then there is a connection ∇^F defined along γ that is characterized by the conditions (i) if γ is geodesic, then $\nabla^F = \gamma^* \nabla^L$, (ii) if $|\dot\gamma|$ is constant, then $\nabla^F_{\mathrm{d}/\mathrm{d}t}\dot\gamma = 0$, (iii) if $W \in \mathcal{X}(\gamma^* TN)$ is orthogonal to $\dot\gamma$ and $\gamma^*\nabla_{\mathrm{d}/\mathrm{d}t}\dot\gamma$, then $\nabla^F_{\mathrm{d}/\mathrm{d}t} W = \gamma^*\nabla^L_{\mathrm{d}/\mathrm{d}t} W$. This connection is known as the *Fermi connection* along γ and has the explicit form

$$\nabla^F_{\frac{\mathrm{d}}{\mathrm{d}t}} W = \gamma^* \nabla^L_{\frac{\mathrm{d}}{\mathrm{d}t}} W + \frac{\sigma(\dot\gamma)}{|\dot\gamma|^2}\left(q(W,\dot\gamma)\gamma^*\nabla^L_{\frac{\mathrm{d}}{\mathrm{d}t}}\dot\gamma - q(W,\gamma^*\nabla^L_{\frac{\mathrm{d}}{\mathrm{d}t}}\dot\gamma)\dot\gamma\right) \tag{2.1}$$

where $\sigma : TN \to \{-1, 1\}$ is given by $\sigma(v) = -\mathrm{sign}(g(v,v))$ for $v \in TN$. In the following we shall suppress pullbacks by γ^*.

This section will show how this formula can be obtained from an inertial connection. First note that nonlinear geometries modeled on the cotangent bundle admit an additional structure. There exists a 1-form α such that $\mathrm{d}\alpha = \omega$ and $\alpha|_X = 0$. The vector field X_α defined by $i(X_\alpha)\omega = \alpha$ has the properties that $X_\alpha \in \mathcal{X}(X)$ and $L_{X_\alpha}\omega = \omega$. Because of the latter property, X_α is called a *homogeneity operator for the triple* $((X,Y), g, \omega)$.

To obtain (2.1) from an inertial connection, ω must be the canonical 2-form and $X = VT^*N$. The problem is to determinate Y and g. The first restriction on Y is that the fiber distance function ρ must be constant along Y. Recall that $\rho \in \mathcal{F}(T^*N)$ is defined by $\rho(p) = |q(p,p)|^{\frac{1}{2}}$ for $p \in T^*N_{\pi(p)}$, and let $\ell : T^*N \to TN$ be the Legendre map defined by the metric. Define $j : VT^*N_p \to TN_{\pi(p)}$ by $j = \ell \circ i$. If $\gamma : \mathbb{R} \to N$, then the natural lift of γ to T^*N is defined by $\tilde\gamma = \ell^{-1}\dot\gamma$.

LEMMA 2.1. *If ρ is constant along Y, then for any $\gamma : \mathbb{R} \to NP^{\perp}\dot{\tilde{\gamma}} = j^{-1}\nabla^{L}_{\frac{d}{dt}}\dot{\gamma}$.*

Proof. Let H be the horizontal distribution of the Levi-Civita connection, and let P_H be the projection onto H. Let Z_O be the metric Hamiltonian vector field. Since ρ is constant on $Y, Z_O \in \mathcal{X}(Y)$. However, $\ell = \pi^* Z_O$ and so $P\dot{\tilde{\gamma}} = Z_O|_{\tilde{\gamma}}$. Since $Z_O \in \mathcal{X}(H)$ it follows that $P\dot{\tilde{\gamma}} = P_H\dot{\tilde{\gamma}}$. ∎

Denote by $\bar{g}$ the affine extension of q to VT^*N. Note that $\bar{g}|_p = j^* q_{\pi(p)}$. Also, if $W \in \mathcal{X}(N)$ denote by $\hat{W}$ the lift of W to Y. In order that an inertial connection induces a metric transport along curves on N, g must be conformal to $\bar{g}$; that is, there is $\varphi \in \mathcal{F}(T^*N)$ such that $g = \varphi\bar{g}$.

PROPOSITION 2.1. *If φ and ρ are constant along Y, then for any $\gamma : \mathbb{R} \to N$ there exist a pair of connections ∇^V and ∇^H along γ such that for $W \in \mathcal{X}(\gamma^*TN)\nabla^V$ and ∇^H are given by*

$$\nabla^{H}_{\frac{d}{dt}} W = \pi^* \nabla_{\frac{d}{dt}} \hat{W} = \nabla^{L}_{\frac{d}{dt}} W + \pi^* T(j^{-1}\nabla^{L}_{\frac{d}{dt}}\dot{\gamma}, \hat{W}), \tag{2.2}$$

$$\nabla^{V}_{\frac{d}{dt}} W = j\nabla_{\frac{d}{dt}} j^{-1} W = \nabla^{L}_{\frac{d}{dt}} W - jS(j^{-1}\nabla^{L}_{\frac{d}{dt}}\dot{\gamma}, j^{-1}W). \tag{2.3}$$

Proof. Extend $\hat{W} \in \mathcal{X}(\tilde{\gamma}^*TT^*N)$ by $\bar{\nabla}$-translation along X to a vector field $\hat{W}$ defined in a neighborthhood of $\tilde{\gamma}$. This is possible since the facts that $d\omega = 0$ and X is integrable imply that the curvature $\bar{R}$ of $\bar{\nabla}$ satisfies $\bar{R}(U,V)Z = 0$ for $U, V \in \mathcal{X}(X)$. Now since $P\dot{\tilde{\gamma}} = \hat{\dot{\gamma}}$,

$$\nabla_{\dot{\tilde{\gamma}}}\hat{W} = \nabla_{\hat{\dot{\gamma}}}\hat{W} + (\nabla - \bar{\nabla})_{P^{\perp}\dot{\tilde{\gamma}}}\hat{W} = \nabla_{\hat{\dot{\gamma}}}\hat{W} + PT(j^{-1}\nabla^{L}_{\frac{d}{dt}}\dot{\gamma}, \hat{W}).$$

But, since φ is constant along $Y, \pi^*\nabla_{\dot{\tilde{\gamma}}}\hat{W} = \nabla^{L}_{\dot{\gamma}} W$. The second identity follows similarly. ∎

In Proposition 2.1 the conformal properties of g were used only to lift the Levi-Civita connection to T^*N. For this construction the most important implication of the conformal condition is that the horizontal component torsion of the inertial connection is determined by φ. Define $\sigma : T^*N \to \{-1, 1\}$ by $\sigma(p) = -\text{sign}(q(p,p))$.

LEMMA 2.2. *If there is $\varphi \in \mathcal{F}(T^*N)$ such that $g = \varphi\bar{g}$, then for $U \in \mathcal{X}(X)JU = \varphi(\widehat{JU})$ and for $U \in \mathcal{X}(Y)JU = \frac{1}{-\varphi}(j^{-1}\pi_* U)$.*

Proof. Note that if $V \in \mathcal{X}(X)$, then $\omega(V,U) = i(V)(\pi_* U)$. Therefore if $U, V \in \mathcal{X}(X)$, then $g(V,U) = \varphi\ell^{-1}(jU)(jV)$ and $\omega(V, JU) = i(V)(\pi_* JU)$, and so $JU = \varphi(\widehat{jU})$. The other identity follows similarly. ∎

PROPOSITION 2.2. *If* ρ *is constant along* Y *and* $g = \rho^k \bar{g}$, *then for any* $\gamma : \mathbb{R} \to N$ *and any* $W \in \mathcal{X}(\gamma^* TN)$

$$PT(j^{-1}\nabla^L_{\frac{d}{dt}}\dot{\gamma}, \hat{W})_{\tilde{\gamma}} = \frac{k\sigma}{2\rho^2}\Big(q(W, \nabla^L_{\frac{d}{dt}}\dot{\gamma}), \hat{\dot{\gamma}} + + q(\dot{\gamma}, \nabla^L_{\frac{d}{dt}}\dot{\gamma})\hat{W} - q(W, \dot{\gamma})\widehat{\nabla^L_{\frac{d}{dt}}\dot{\gamma}}\Big) \tag{2.4}$$

$$S(j^{-1}\nabla^L_{\frac{d}{dt}}\dot{\gamma}, j^{-1}W)_{\tilde{\gamma}} = \frac{k\sigma}{2\rho^2}\Big(q(\dot{\gamma}, \nabla^L_{\frac{d}{dt}}\dot{\gamma})j^{-1}W + + q(W, \dot{\gamma})j^{-1}\nabla^L_{\frac{d}{dt}}\dot{\gamma} - q(w, \nabla^L_{\frac{d}{dt}}\dot{\gamma})\, j_{-1}\dot{\gamma}\Big) \tag{2.5}$$

Proof. First note that if α is the canonical 1-form, then X_α is a homogeneity operator. Since $\bar{\nabla}$ is metric for $\bar{g}$ along X, and g is conformal to $\bar{g}$, it follows that for $U, V \in \mathcal{X}(X)$

$$S(U,V) = \frac{k\sigma}{2\rho^2}\left(\bar{g}(X_\alpha, U)V + \bar{g}(V, X_\alpha)U - \bar{g}(U,V)X_\alpha\right)$$

Now by proposition 1.4, it follows that for $U \in \mathcal{X}(Y)$ and $V \in \mathcal{X}(X)$

$$PT(U,V) = \frac{k\sigma}{2\rho^2}\big(\bar{g}(X_\alpha, JU)JV + \bar{g}(V, X_\alpha)U - - \bar{g}(JU, V)JX_\alpha\big)$$

Since $X_\alpha|_{\tilde{\gamma}} = j^{-1} \cdot \gamma$ (2.4) and (2.5) now follow from Lemma 2.2. ∎

By comparing (2.1), (2.4) and (2.5) it is easily seen that proposition 2.1 implies the following.

PROPOSITION 2.3. *If* $g = \rho^{-2\sigma}\bar{g}$, *then* $\nabla^F = (\nabla^V + \nabla^H)/2$. ∎

Along curves of constant length it is seen that $\nabla^V = \nabla^H$, and so in this case both ∇^V and ∇^H give the Fermi connection. An inertial connection can be defined so that $\nabla^H = \nabla^F$, but the torsion does not satisfy the propositions of section 1. Also note that the fact that $\nabla^H = \nabla^V = \nabla^F$ along unit curves implies that lifted curves which are geodesics of the inertial connection are just those curves with Fermi parallel acceleration.

SECTION 3. WEYLIAN ELECTROMAGNETISM

In the last section it was shown that certain inertial connections on T^*N determine connections along curves on N, and further that the Fermi connection is among these.

Since Fermi transport represents the effects of special relativity in the frame of an accelerated observer, it is natural to ask if the principles of special relativity can be formulated in terms of inertial connections. Here we present two criteria that determine Fermi transport up to a scale and a choice of zero section. These criteria extend to manifolds the two postulates of special relativity that state (i) Galilean frames are physically indistinguishable and (ii) the speed of light is an invariant of all Galilean frames. The first postulate expresses a homogeneity condition on quantities observed in a frame. However, if this principle is extended to manifolds, it can only be required to hold infinitesimally and only for simultaneous observers. Recall that two observers represented by $p, q \in M$ are *simultaneous* if p and q lie in the same leaf L of X. The inertial and Bott connection determine a singular G-bundle G over L that is associated with the vector bundle $Y|_L$. For each $p \in L, G_p$ is the smallest Lie subgroup of $Gl(Y_p)$ that contains the endomorphisms of Y_p generated by choosing a point $q \in L$ and composing ∇-translation along the affine geodesic λ from p to q with $\bar{\nabla}$-translation along λ from q to p. G_p can be interpreted as the group of infinitesimal kinematic transformations betwenn p and simultaneous observers in neighboring frames. If $M = T^*N$ and ∇ gives Fermi translation, then $G_p = Cso(Y_p)$, the group of linear conformal maps of Y_p. If $\mathfrak{g}_p$ is the Lie algebra of G_p, then by the remark before Proposition 1.4 it follows that $\mathfrak{g}_p$ is generated by the image of the map $PT : X_p \to \mathrm{End}(Y_p)$ which is given by $PT(v)(u) = PT(v, u)$ for $v \in X_p$ and $u \in Y_p$.

DEFINITION 3.1. $((X, Y), g, \omega)$ is *homogeneous* if there exists $\kappa \in \mathbb{R}$ such that for $U, V \in \mathcal{X}(X)$ and $W \in \mathcal{X}(X)$

$$\kappa R(U, V)W = [PT(U), PT(V)]W \tag{3.1}$$

κ is called the *scale* of the homogeneous geometry $((X, Y), g, \omega)$.

Definition 3.1 gives the simplest condition that guarantees that the curvature of an inertial geometry takes values in the Lie algebra bundle of G. It implies that parallel translation induces an isomorphism of the fibers of G. This equivalence of the infinetisimal kinematic symmetric groups gives a representation of postulate (i). Also note that (3.1) also gives a representation of Thomas precession in terms of curvature.

Postulate (ii) has a more direct geometric representation.

DEFINITON 3.2. $((X, Y), g, \omega)$ is *semi-metric* if there exists $\lambda \in \mathcal{E}^1(M)$ such that $Y \subset \ker(\lambda)$ and for $V \in \mathcal{X}(X)$ and $U, W \in \mathcal{X}(Y)$

$$\bar{\nabla}_V g(U, W) = \lambda(V) g(U, V). \tag{3.2}$$

Definition 3.2 requires that $\bar{\nabla}$-parallel translation along X be a conformal transformation of the metric along Y. Since conformal transformations preserve light cones and since $\bar{\nabla}$ determines the affine structure of the complete set of frames determined by Y, definition 3.2 states that for simultaneous observers the light cone is affine invariant. The following propositions develop the consequences of definitions 3.1 and 3.2.

LEMMA 3.1. *If $((X,Y),g,\omega)$ is semi-metric, then $((X,Y),g,\omega)$ is homogeneous if and only if for $U,V \in \mathcal{X}(X)$*

$$\nabla_U\lambda(V) - \frac{\kappa-1}{2\kappa}\lambda(U)\lambda(V) + \frac{\kappa-1}{4\kappa}|\lambda|^2 g(U,V) = 0. \tag{3.3}$$

Proof. Since $\bar{\nabla}\omega = 0$ it follows that for $U \in \mathcal{X}(Y)$ and $V \in \mathcal{X}(X), \bar{\nabla}J(U) = \lambda(V)JU$. But, since $J(\bar{\nabla}J)+(\bar{\nabla}J)J = 0$, this implies for $U,V \in \mathcal{X}(X)\bar{\nabla}_V J(U) = \lambda(V)JU$, and consequently for $U,V,W \in \mathcal{X}(X)\bar{\nabla}_V g(U,W) = \lambda(V)g(U,W)$. Since ∇ is Levi-Civita and $\bar{\nabla}$ is torsion free (3.2) implies

$$\begin{aligned} g(U,S(V,W)) &= \frac{1}{2}(\lambda(W)g(U,V) + \lambda(V)g(U,W) - \\ &\quad -\lambda(U)g(V,W)). \end{aligned} \tag{3.4}$$

Also proposition 1.3 and 1.4 imply that (3.1) reduces to

$$\begin{aligned} &\nabla_U S(V,W) - \nabla_V S(U,W) + \left(\frac{\kappa-1}{\kappa}\right)(S(U,S(V,W)) - \\ &\quad - S(V,S(U,W))) = 0. \end{aligned} \tag{3.5}$$

Since $\bar{\nabla}$ is flat along $X, \lambda|_X$ is closed and so (3.4) and (3.5) imply that (3.3) is equivalent to (3.1). ∎

Lemma 3.1 has a simpler expression in terms of radial vector fields. If D is a connection, then a vector field X is $\mathrm{D}-radial$ if the endomorphism $\mathrm{D}X$ is a multiple of the identity; that is, there is $r \in \mathbb{R} - \{0\}$ such that $\mathrm{D}X = rI$. If λ is a form or vector field, $\check{\lambda}$ be the metric dual of λ.

LEMMA 3.2. *If $((X,Y),g,\omega)$ is semi-metric, then $((X,Y),g,\omega)$ is homogeneous if and only if $X_\lambda = \check{\lambda}/|\lambda|^2$ is ∇-radial and $r \neq \frac{1}{4}$,* ∎

This lemma now allows a complete characterization of homogeneous, semi-metric nonlinear geometries.

PROPOSITION 3.1. *$((X,Y),g,\omega)$ is homogeneous and semi-metric if and only if there exists an affine metric $\bar{g}$ along X and $\bar{\nabla}$-radial vector field $R \in \mathcal{X}(X)$ such that g is conformal to $\bar{g}$ and $L_R g = cg$ for some $c \in \mathbb{R}$.*

Proof. Suppose that $((X,Y),g,\omega)$ is semi-metric and homogeneous. Clearly, g is conformal to an affine metric; in fact, $\bar{g} = \left(|g(X_\lambda, X_\lambda)|^{\frac{2\kappa}{3\kappa+1}}\right) g$ is affine. Also since X_λ is ∇-radial and $S(X_\lambda, U) = \frac{1}{2}U$ for $U \in \mathcal{X}(X)$, it follows that $\bar{\nabla}_U X_\lambda = \frac{\kappa+1}{4\kappa}U$. To see the other implication, it may be assumed that $\bar{\nabla}R = I$. Since the leaves of X are connected, R has one and only one zero on each leaf of X. Let $\rho : M \to \mathbb{R}$ be the distance function along X to the zero of R. The hypothesis on g implies that $g = \rho^k \bar{g}$, and so $L_R g = (k+2)g$. Further, $\mathrm{d}\,R = 0$ and so $g(\nabla_U R, V) = \frac{1}{2}L_R g(U,V) = \left(\frac{k+2}{2}\right) g(U,V)$. Therefore, R is ∇-radial, and since $((X,Y),g,\omega)$ is semi-metric, lemma 3.1 implies homogeneity. ■

Note that the proof of proposition 3.1 shows that Fermi transport is obtained when $\kappa = 1$ and $2X_\lambda = X_\alpha$.

We now show that the freedom to choose R and the scale κ can be related to the electromagnetic field and the electric charge. This interpretation will follow from the standard symplectic representation of the electromagnetic field; see [4]

DEFINITION 3.3. A closed 2-form ω' satisfies a *Maxwell condition* for $((X,Y),g,\omega)$ if there is $e \in \mathbf{R} - \{0\}$ such that for all $V \in \mathcal{X}(X)$ (i) $i(V)(\omega' - e\omega) = 0$ and (ii) $\bar{\nabla}_V \omega' = 0$.

In the case of the cotangent bundle this condition guarantees that the spray of the Hamiltonian vector field Z' defined by ω' and the metric determines a Lorentz force law. If X_α is a homogeneity operator for ω then, in terms of the present notation, Z' is given by

$$ei(X_\alpha)\bar{g} = i(Z')\omega'. \tag{3.6}$$

Here $\bar{g}$ is the extension to T^*N of the affine metric along X. If the distance function ρ is constant along Y, then $\ell_* Z'$ is the spray for a Lorentz force law with field strength F given by $\pi^* F = \omega' - e\omega$. Note that e is the inverse of the charge. In this construction e is most easily interpreted as a dimensionless scaling factor. The next proposition shows that the dilation λ is related to the potential of a 2-form that satisfies definition 3.3.

PROPOSITION 3.2. *For a homogeneous and semi-metric nonlinear geometry let* $\beta = i(J)\lambda/|\lambda|^2$. *Then* $\omega' = \mathrm{d}\,\beta$ *satisfies a Maxwell condition with* $e = \frac{\kappa+1}{4\kappa}$.

Proof. Note that $\beta = i(X_\lambda)\omega$. The fact that X_λ is $\bar{\nabla}$-radial implies definition 3.3 (i). This fact and the fact that $\mathrm{d}\,\omega' = 0$ imply definition 3.3 (ii). ■

To see the relation between β and the standard representation of the vector potential note that $R = (\frac{4\kappa}{\kappa+1})X_\lambda$ satisfies $\bar{\nabla}R = I$. Therefore the vector field $a = (\frac{\kappa+1}{4\kappa})X_\alpha -$

X_λ is a $\bar{\nabla}$-parallel field, and consequently $\omega' = (\frac{\kappa+1}{4\kappa})\omega + d\, i(a)\omega$. In the case of the cotangent bundle, the vector field $(\frac{4\kappa}{\kappa+1})a$ represents the affine translation needed to take the standard radial field X_α on to the radial field defined by the nonlinear geometry with dilation λ.

What has been shown is that definitions 3.1 and 3.2 determine a class of nonlinear geometries that differ from the geometry used to obtain Fermi transport by a choice of scale and a choice of origin. Further, these degrees of freedom can be related to the mechanical aspects of the electromagnetic field. There are similarities and differences between this constructions and Weyl's theory. The most significant difference is that in this construction the connection to which the semi-metric condition applies is not the dynamical space-time connection, that is, definition 3.2 determines the metric g and not the connection $\bar{\nabla}$. Both costrunctions have a similar motivation for applyng the semi-metric condition. However, in Weyl's theory the scale variable that arises from the semi-metric connection is hard to interpret and leads to physical inconsistencies; see [1]. In contrast, when the semi-metric condition is employed as in definition 3.2 and used in conjunction with definition 3.1, the scale appears as a real number and has an immediate electromagnetic interpretation; namely it is related to the charge. In fact, according to this construction, because the charge appears in definition 3.1 it is more closely associated with the kinematic structure of space-time than with the field structure. This fact could provide an explanation for the uniqueness of the electric charge. Note that if $a = 0$, then the charge cancels in (3.6) leaving the geodesic equation of free space. Both Weyl's theory and this construction make new predictions about the nature of space-time. In this construction, the relation between inertial frames need not be the affine relation that is customarily and tacitly but rather a scaling relation.

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